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THE INSTABILITY OF SLIGHTLY COMPRESSIBLE RECTANGULAR RUBBERLIKE SOLIDS UNDER BIAXIAL LOADINGS†

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Abstract—The instability of a rectangular solid loaded in plane strain by constant axial and transverse pressures is investigated using a variational method. Axial pressures are applied through rigid platens, and lateral pressures can be either hydrostatic or constant-directional. Two types of slightly compressible rubberlike material are studied in the problem, both of which reduce to Rivlin's neo-Hookean material in the incompressible case. Previous studies using a "standard" material and the neo-Hookean material have yielded qualitatively different results.

It is found that the "standard" material is unsuitable for studies of such finite strain problems and that the behaviour of the compressible materials studied tends smoothly towards that of the corresponding incompressible material as Poisson's ratio approaches one half.

1. INTRODUCTION

THE instability of a rectangular solid in plane strain, loaded axially by a constant pressure p and laterally by a constant pressure q was studied by Kerr and Tang [1–3], using constitutive relations derived from the "standard" strain energy density of John [4]. They found, considering the effect of two different types of side-pressure, that a hydrostatic pressure effectively stabilized the equilibrium of the solid, while a constant-directional pressure destabilized it. Wu and Widera [5] pointed out that the "standard" strain energy density does not give a good description of the behaviour of real materials other than in situations of infinitesimal strain and that the problem considered necessarily involves large strains. They therefore re-studied the problem using the Mooney–Rivlin constitutive relation for incompressible rubberlike materials, a special case of which is what Rivlin [6] calls the neo-Hookean constitutive relation. Wu and Widera's studies made use of the general nonlinear tensorial treatment given by Green and Zerna [7]. Their conclusions were that contrary to the findings of [1–3] both hydrostatic and constant-directional pressures stabilized the solid. They concluded also that the stabilizing effect of constant-directional side-pressure was greater for thinner solids, while that due to hydrostatic side-pressure was independent of the shape of the solid. Both asymmetric (flexural) and symmetric (bulging) instability were considered, and the same conclusions were found to be applicable in each case.

The present authors were moved by the qualitatively different nature of the results of [1–3] and of [5] to investigate whether these differences were caused by the precise natures of the constitutive relations used, or by the constraint of incompressibility in [5]. In the course

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of the work it was discovered, and pointed out [8], that Wu and Widera's transcendental equations for the stability bounds contained a trivial numerical error which had been carried over into the graphical presentation of their results, qualitatively affecting their conclusions in a non-trivial manner (as will be shown later in this paper).

In the present paper the problem is re-studied using two relatively simple constitutive relations for slightly compressible rubberlike materials examined in [9], both of which reduce to the neo-Hookean material in the incompressible limit. These constitutive relations most closely represent realistic material behaviour when Poisson's ratio is close to 0·5 as is shown in some detail in [9]. Consequently, the ideal materials described by such relations will be called *slightly* compressible materials. These are the Blatz–Ko [10] material and a material with a simpler constitutive relation, called the compressible polynomial material, which was introduced in [9]. The analysis is carried out using a variational approach developed [11] from the earlier work of Hill [12] and Pearson [13] on the stability of a state of finite strain of a hyperelastic continuum. The materials considered are defined in terms of the J invariants of deformation used by Blatz and Ko. These are related to the usual I invariants by the relations

$$J_1 = I_1, \qquad J_2 = \frac{I_2}{I_3}, \qquad J_3 = (I_3)^{\frac{1}{2}}, \tag{1}$$

where

$$\begin{cases} I_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2, \\ I_2 = \lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2, \\ I_3 = \lambda_1^2\lambda_2^2\lambda_3^2, \end{cases} \tag{2}$$

and $\lambda_1, \lambda_2, \lambda_3$ are the principal extension ratios.

The second invariant J_2 does not appear in the specific constitutive relations used. The two compressible materials chosen tend to act as foils for one another since they have qualitatively different behaviour, the Blatz–Ko material tending to soften in tension and harden in compression, and the polynomial material having the opposite tendencies.

2. EQUATIONS OF NEUTRAL EQUILIBRIUM

This section contains a re-statement of some of the results of [11] in a form more suitable for the present investigation. The system considered is a hyperelastic body of arbitrary shape. The body is initially unloaded (state I), and the rectangular coordinates of a particle P in the body are (a_1, a_2, a_3). The body is loaded by surface tractions alone and deforms to an equilibrium state II, when the deformed coordinates of P are (x_1, x_2, x_3). The surface tractions are assumed to be conservative and non-discrete. The stress components of the applied surface tractions are denoted by P_{ij}, these being the boundary values of the internal stress components τ_{ij}. The (finite) components of the displacement of the particle P are given by

$$v_i = x_i - a_i, \tag{3}$$

and the Lagrangian strain tensor is

$$\eta_{ij} = \frac{1}{2}\left[\frac{\partial v_i}{\partial a_j}+\frac{\partial v_j}{\partial a_i}+\frac{\partial v_s}{\partial a_i}\frac{\partial v_s}{\partial a_j}\right]. \tag{4}$$

In state II the stress components at the point occupied by P are denoted as τ_{ij}, the "true" stress, the strain energy of deformation is U per unit mass and the density of the body is ρ (cf. ρ_0 in state I).

To investigate equilibrium and stability in state II infinitesimal increments u_i are superimposed upon the finite displacement components v_i. It is to be noted that in the special case of an incompressible material the u_i must satisfy the linear incompressibility condition.

The vanishing of the first variation of the total potential energy under these perturbations u_i gives the constitutive equations in state II. For the sake of completeness these equations are written so as to include the limiting case of incompressibility.

$$\tau_{ij} = \rho\left(\frac{\partial U}{\partial \eta_{pq}}\right)\frac{\partial x_i}{\partial a_p}\frac{\partial x_j}{\partial a_q} + \sigma\delta_{ij} \tag{5}$$

where σ is a Lagrange multiplier which vanishes identically for any compressible material, and δ_{ij} is the Kronecker delta. For an incompressible material σ exists and the equilibrium equations must be solved together with the incompressibility condition.

From this point the following notation is used:

$$u_{i,j} = \frac{\partial u_i}{\partial x_j}; \qquad \sigma_{,k} = \frac{\partial \sigma}{\partial x_k}; \qquad u_{i,jk} = \frac{\partial^2 u_i}{\partial x_j \partial x_k}; \qquad \text{etc.} \tag{6}$$

The equations of neutral equilibrium are now investigated under the assumption that surface tractions are conservative and nondiscrete. The infinitesimal strain tensor for perturbations on state II is given by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{7}$$

and the tensor γ_{ijkl} is defined (after Hill [12]) as

$$\gamma_{ijkl} = \frac{\rho}{\rho_0}\left(\frac{\partial^2 U}{\partial \eta_{pq}\partial \eta_{mn}}\right)\frac{\partial x_i}{\partial a_p}\frac{\partial x_j}{\partial a_q}\frac{\partial x_k}{\partial a_m}\frac{\partial x_l}{\partial a_n}. \tag{8}$$

The equations of neutral equilibrium consist of three differential equations (d.e.'s) holding at every point of the deformed body and of three boundary conditions (b.c.'s) which hold on each piecewise-continuous portion of the boundary surface. The three d.e.'s are given by

$$\begin{aligned} &-\sigma'_{,i} + \sigma u_{i,jj} + 2\sigma_{,j}\varepsilon_{ij} - \tau_{jk}u_{i,jk} - \tau_{jk,k}u_{i,j} \\ &-\gamma_{ijkl}u_{k,jl} - \gamma_{ijkl,j}u_{k,l} = 0; \qquad i = 1, 2, 3. \end{aligned} \tag{9}$$

Here σ' is another Lagrange multiplier connected with incompressibility in the neutral equilibrium state, vanishing identically for compressible materials. The b.c.'s over the piecewise-continuous boundary surface A are given by

$$\int_A w_j\, \mathrm{d}A_i[\sigma'\delta_{ij} - 2\sigma\varepsilon_{ij} + \gamma_{ijkl}u_{k,l} + p_{ik}u_{j,k} - p_{ij}u_{k,k} + p_{kj}u_{i,k} - p_{ij,k}u_k] = 0 \tag{10}$$

for $j = 1, 2, 3$. The component w_j is the increment δu_j and $\mathrm{d}A_i = (\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3)/\mathrm{d}x_i$. In the incompressible case the linear incompressibility condition $u_{i,i} = 0$ must hold throughout the body.

3. THE PROBLEM OF A BIAXIALLY LOADED RECTANGULAR PRISM

The problem concerns the stability of a rectangular elastic solid of infinite extent in the a_3 direction loaded in plane strain by a lateral (compressive) pressure q and an axial (compressive) pressure p which is applied through rigid frictionless platens maintained at a fixed distance from each other. Plane strain implying, of course, that $\eta_{3j} = 0, j = 1, 2, 3$. The nature of q can be either hydrostatic or constant-directional (i.e. the action of q does not change direction during any deformation of the body). Both cases involve conservative loadings. The second case is obvious and the first is so because of the fixed distance between the platens. The argument is given in [3]. It is apparent that certain simplifications occur in the general equations of equilibrium and of neutral equilibrium as follows:

(i) If the material of the body is compressible, σ, σ' and all of their derivatives vanish.

(ii) Since the deformation from state I to state II was homogeneous the internal stress components in state II are $\tau_{11} = -p$, $\tau_{22} = -q$, $\tau_{ij} = 0$ if $i \neq j$.

(iii) On the boundaries parallel to the x_1-axis $p_{22} = -q$, $p_{11} = -\alpha q$ where $\alpha = 1$ for hydrostatic pressure and $\alpha = 0$ for constant-directional pressure. Also $dA_1 = dA_3 = 0$.

(iv) On the boundaries parallel to the x_2-axis (acted on by the rigid platens) $p_{11} = -p$, $p_{22} = 0$. Also $dA_2 = dA_3 = 0$.

It is easily shown that the deformation invariants J_1 and J_3 in state II are given by

$$J_1 = \lambda_1^2+\lambda_2^2+\lambda_3^2 = 3+2\eta_{11}+2\eta_{22}, \tag{11}$$

and

$$J_3^2 = \lambda_1^2\lambda_2^2\lambda_3^2 = (1+2\eta_{11})(1+2\eta_{22})-4\eta_{12}^2 \tag{12}$$

since it is assumed $\lambda_3 = 1$. Note that $\eta_{12} = \eta_{21}$ so that η_{21} has been included above. Hence, the relevant derivatives of J_3 may be tabulated as follows in Table 1 below, making use of the homogeneous nature of the deformation and equation (4).

4. SOLUTION FOR THE COMPRESSIBLE POLYNOMIAL MATERIAL

The constitutive relation for the compressible polynomial material is

$$U = \frac{\mu}{2}[(J_1-3)-2(J_3-1)+B(J_3-1)^2] \tag{13}$$

in which $B = 1/(1-2\nu)$. μ is the shear modulus, and ν Poisson's ratio for the undeformed body. Let $A = [2(1-\nu)]/(1-2\nu)$ in order that the relevant derivatives of U may be written as follows:

$$\frac{\partial U}{\partial \eta_{11}} = \mu\left[1-\frac{A\lambda_2}{\lambda_1}+B\lambda_2^2\right], \tag{14}$$

$$\frac{\partial U}{\partial \eta_{22}} = \mu\left[1-\frac{A\lambda_1}{\lambda_2}+B\lambda_1^2\right], \tag{15}$$

$$\frac{\partial U}{\partial \eta_{12}} = 0, \tag{16}$$

TABLE 1

Derivative	General expression	Evaluation at state II
$\dfrac{\partial J_3}{\partial \eta_{11}}$	$\dfrac{1}{J_3}(1+2\eta_{22})$	λ_2/λ_1
$\dfrac{\partial J_3}{\partial \eta_{22}}$	$\dfrac{1}{J_3}(1+2\eta_{11})$	λ_1/λ_2
$\dfrac{\partial J_3}{\partial \eta_{12}}$	$-\dfrac{4\eta_{12}}{J_3}$	0
$\dfrac{\partial^2 J_3}{\partial \eta_{11}^2}$	$-\dfrac{1}{J_3^3}(1+2\eta_{22})^2$	$-\lambda_2/\lambda_1^3$
$\dfrac{\partial^2 J_3}{\partial \eta_{22}^2}$	$-\dfrac{1}{J_3^3}(1+2\eta_{11})^2$	$-\lambda_1/\lambda_2^3$
$\dfrac{\partial^2 J_3}{\partial \eta_{11}\,\partial \eta_{22}}$	$\dfrac{2}{J_3}-\dfrac{1}{J_3^3}(1+2\eta_{11})(1+2\eta_{22})$	$1/\lambda_1\lambda_2$
$\dfrac{\partial^2 J_3}{\partial \eta_{12}^2}$	$-\dfrac{4}{J_3}-\dfrac{16}{J_3}(\eta_{12})^2$	$-4/\lambda_1\lambda_2$
$\dfrac{\partial^2 J_3}{\partial \eta_{11}\,\partial \eta_{12}}$	$\dfrac{1}{J_3^3}(1+2\eta_{22})(4\eta_{12})$	0
$\dfrac{\partial^2 J_3}{\partial \eta_{22}\,\partial \eta_{12}}$	$\dfrac{1}{J_3^3}(1+2\eta_{11})(4\eta_{12})$	0

$$\frac{\partial^2 U}{\partial \eta_{11}^2} = \mu \frac{A\lambda_2}{\lambda_1^3}, \tag{17}$$

$$\frac{\partial^2 U}{\partial \eta_{22}^2} = \mu \frac{A\lambda_1}{\lambda_2^3}, \tag{18}$$

$$\frac{\partial^2 U}{\partial \eta_{11}\partial \eta_{22}} = \mu\left[-\frac{A}{\lambda_1\lambda_2}+2B\right], \tag{19}$$

$$\frac{\partial^2 U}{\partial \eta_{12}^2} = 4\mu\left[\frac{A}{\lambda_1\lambda_2}-B\right]. \tag{20}$$

Taking into account that $\eta_{12} = \eta_{21}$ the components γ_{ijkl} may now be written:

$$\begin{cases} \gamma_{1111} = \gamma_{2222} = \mu A, & (21)\\ \gamma_{1122} = \gamma_{2211} = \mu[2B\lambda_1\lambda_2 - A], & (22)\\ \gamma_{1212} = \gamma_{1221} = \gamma_{2112} = \gamma_{2121} = \mu[A - B\lambda_1\lambda_2]. & (23) \end{cases}$$

All other $\gamma_{ijkl} = 0$.

The stress–principal extension relationship is given by the equations

$$\begin{cases} -p = \mu\left[\dfrac{\lambda_1}{\lambda_2} - A + B\lambda_1\lambda_2\right], & (24) \\ -q = \mu\left[\dfrac{\lambda_2}{\lambda_1} - A + B\lambda_1\lambda_2\right]. & (25) \end{cases}$$

which follow from equation (5) when $\sigma = 0$.

The specific equations of neutral equilibrium may now be written. The general d.e.'s (9) become, in this case,

$$\begin{cases} u_{1,11}(p-\gamma_{1111})+u_{1,22}(q-\gamma_{1212})-u_{2,12}(\gamma_{1122}+\gamma_{1221}) = 0, & (26) \\ u_{2,11}(p-\gamma_{2121})+u_{2,22}(q-\gamma_{2222})-u_{1,12}(\gamma_{2211}+\gamma_{2112}) = 0, & (27) \end{cases}$$

which may be expressed as

$$\begin{cases} u_{1,11}(p-\mu A)+u_{1,22}(q-\mu A+\mu B\lambda_1\lambda_2)-u_{2,12}\mu B\lambda_1\lambda_2 = 0, & (28) \\ u_{2,11}(p-\mu A)+\mu B\lambda_1\lambda_2)+u_{2,22}(q-\mu A)-u_{1,12}\mu B\lambda_1\lambda_2 = 0. & (29) \end{cases}$$

Equations (28) and (29) may be written using equations (24) and (25), as

$$u_{1,11}\left(\frac{\lambda_1}{\lambda_2}+B\lambda_1\lambda_2\right)+u_{1,22}\left(\frac{\lambda_2}{\lambda_1}\right)+u_{2,12}B\lambda_1\lambda_2 = 0 \tag{30}$$

and

$$u_{2,11}\left(\frac{\lambda_1}{\lambda_2}\right)+u_{2,22}\left(\frac{\lambda_2}{\lambda_1}+B\lambda_1\lambda_2\right)+u_{1,12}B\lambda_1\lambda_2 = 0. \tag{31}$$

The b.c.'s given by equation (10) become:

(i) *On edges* $x_2 = \pm\lambda_2 h/2$

$$[u_{1,2}(\gamma_{2112}-q)+u_{2,1}(\gamma_{2121}-\alpha q)]w_1 = 0, \tag{32}$$

and

$$[u_{1,1}(\gamma_{2211}+q)+u_{2,2}(\gamma_{2222}-q)]w_2 = 0, \tag{33}$$

which may be written

$$u_{1,2}\left(\frac{\lambda_2}{\lambda_1}\right)+u_{2,1}\left(\frac{\lambda_2}{\lambda_1}+\frac{q}{\mu}(1-\alpha)\right) = 0, \tag{34}$$

and

$$u_{1,1}\left(B\lambda_1\lambda_2-\frac{\lambda_2}{\lambda_1}\right)+u_{2,2}\left(B\lambda_1\lambda_2+\frac{\lambda_2}{\lambda_1}\right) = 0. \tag{35}$$

(ii) *On ends* $x_1 = 0, \lambda_1 l$

$$w_1[(u_{1,1}(\gamma_{1111}-p)+u_{2,2}(\gamma_{1122}+p)] = 0, \tag{36}$$

and

$$w_2[u_{2,1}(\gamma_{1221}-p)+u_{1,2}\gamma_{1212}] = 0. \tag{37}$$

Because of the platens through which p is applied w_1 must vanish identically, as must $u_{1,2}$. Hence these b.c.'s reduce to

$$u_{2,1} = u_1 = 0. \tag{38}$$

Displacements are therefore assumed to have representations in the following forms which identically satisfy the boundary conditions (38);

$$u_1 = \sum_{n=1}^{\infty} F_n(\omega_n x_2)\sin\omega_n x_1, \tag{39}$$

$$u_2 = \sum_{n=1}^{\infty} G_n(\omega_n x_2)\cos\omega_n x_1, \tag{40}$$

where $\omega_n = n\pi/\lambda_1 l$.

Simultaneous, ordinary d.e.'s governing F_n and G_n are found by substituting equations (39) and (40) into equations (30) and (31).

$$\left(\frac{\lambda_2}{\lambda_1}\right)F_n''-\left(\frac{\lambda_1}{\lambda_2}+B\lambda_1\lambda_2\right)F_n-B\lambda_1\lambda_2G_n' = 0, \tag{41}$$

$$\left(\frac{\lambda_2}{\lambda_1}+B\lambda_1\lambda_2\right)G_n''-\left(\frac{\lambda_1}{\lambda_2}\right)G_n+B\lambda_1\lambda_2F_n' = 0, \tag{42}$$

where the $'$ superscript denotes a derivative with respect to $\omega_n x_2$.

The general solutions to (41) and (42) are,

$$F_n = C_{1n}\sinh\left(\frac{\lambda_1}{\lambda_2}\omega_n x_2\right)+C_{2n}\cosh\left(\frac{\lambda_1}{\lambda_2}\omega_n x_2\right)+C_{3n}\sinh\left(K\frac{\lambda_1}{\lambda_2}\omega_n x_2\right)+C_{4n}\cosh\left(K\frac{\lambda_1}{\lambda_2}\omega_n x_2\right), \tag{43}$$

and

$$G_n = -\left[\frac{\lambda_2}{\lambda_1}\left(C_{1n}\cosh\left(\frac{\lambda_1}{\lambda_2}\omega_n x_2\right)+C_{2n}\sinh\left(\frac{\lambda_1}{\lambda_2}\omega_n x_2\right)\right)+\frac{K\lambda_1}{\lambda_2}\left(C_{3n}\cosh\left(\frac{K\lambda_1}{\lambda_2}\omega_n x_2\right)+C_{4n}\sinh\left(\frac{K\lambda_1}{\lambda_2}\omega_n x_2\right)\right)\right], \tag{44}$$

where $K = [(1+B\lambda_2^2)/(1+B\lambda_1^2)]^{\frac{1}{2}}$.

The following notation is adopted for the sake of simplicity:

$$\delta_1 = \frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}+(1-\alpha)q, \tag{45}$$

$$\delta_2 = \frac{2K\lambda_1}{\lambda_2}+(1-\alpha)qK\left(\frac{\lambda_1}{\lambda_2}\right)^2, \tag{46}$$

$$\varepsilon_1 = -2, \tag{47}$$

$$\varepsilon_2 = -\left(\frac{\lambda_1^2+\lambda_2^2}{\lambda_2^2}\right), \tag{48}$$

$$\psi_{1n} = \lambda_1\omega_n\frac{h}{2}, \tag{49}$$

$$\psi_{2n} = K\lambda_1\omega_n\frac{h}{2}. \tag{50}$$

ε_1 is included here merely for the sake of notational symmetry.

The general solutions (43) and (44) are now inserted into the b.c.'s (34), (35) whereupon four simultaneous linear equations for the C_{in} are obtained. The vanishing of the determinant of the coefficients of the C_{in} in these equations gives the nth stability bound as follows:

$$\begin{vmatrix} \delta_1\cosh\psi_{1n} & \delta_1\sinh\psi_{1n} & \delta_2\cosh\psi_{2n} & \delta_2\sinh\psi_{2n} \\ \delta_1\cosh\psi_{1n} & -\delta_1\sinh\psi_{1n} & \delta_2\cosh\psi_{2n} & -\delta_2\sinh\psi_{2n} \\ \varepsilon_1\sinh\psi_{1n} & \varepsilon_1\cosh\psi_{1n} & \varepsilon_2\sinh\psi_{2n} & \varepsilon_2\cosh\psi_{2n} \\ -\varepsilon_1\sinh\psi_{1n} & \varepsilon_1\cosh\psi_{1n} & -\varepsilon_2\sinh\psi_{2n} & \varepsilon_2\cosh\psi_{2n} \end{vmatrix} = 0. \tag{51}$$

This may be rearranged to give

$$\begin{vmatrix} \delta_1\cosh\psi_{1n} & \delta_2\cosh\psi_{2n} & 0 & 0 \\ \varepsilon_1\sinh\psi_{1n} & \varepsilon_2\sinh\psi_{2n} & 0 & 0 \\ 0 & 0 & \delta_1\sinh\psi_{1n} & \delta_2\sinh\psi_{2n} \\ 0 & 0 & \varepsilon_1\cosh\psi_{1n} & \varepsilon_2\cosh\psi_{2n} \end{vmatrix} = 0, \tag{52}$$

or $|\Delta_1| \,.\, |\Delta_2| = 0$ where $|\Delta_1|$ and $|\Delta_2|$ are the two non-zero minors appearing in equation (52).

The two solutions to this equation represent symmetric and asymmetric deformation about the axis of the deformed body. These modes of deformation will be referred to as bulging and bending (or flexing), respectively. The solutions can be put into a form similar to that given by Wu and Widera for the incompressible case as follows:

$$\frac{1}{\mu}(p-q) = \frac{2\lambda_2}{\lambda_1} - \left[\frac{2K\lambda_1\lambda_2\left(2+(1-\alpha)\frac{q}{\mu}\frac{\lambda_1}{\lambda_2}\right)}{\lambda_1^2+\lambda_2^2+(1-\alpha)\frac{q}{\mu}\lambda_1\lambda_2}\right]\tanh\left(\frac{n\pi h}{2l}\right)\coth\left(\frac{Kn\pi h}{2l}\right) \tag{53}$$

for flexural instability, and

$$\frac{1}{\mu}(p-q) = \frac{2\lambda_2}{\lambda_1} - \left[\frac{2K\lambda_1\lambda_2\left(2+(1-\alpha)\frac{q}{\mu}\frac{\lambda_1}{\lambda_2}\right)}{\lambda_1^2+\lambda_2^2+(1-\alpha)\frac{q}{\mu}\lambda_1\lambda_2}\right]\coth\left(\frac{n\pi h}{2l}\right)\tanh\left(\frac{Kn\pi h}{2l}\right) \tag{54}$$

for bulging instability.

5. SOLUTION FOR THE BLATZ–KO MATERIAL

The material suggested by Blatz and Ko (special case $f = 1$) has the constitutive relation

$$U = \frac{\mu}{2}\left[(J_1 - 3) + \left(\frac{1-2\nu}{\nu}\right)(J_3^{-2\nu/(1-2\nu)} - 1)\right] \tag{55}$$

for the case of a Blatz–Ko material independent of J_2.

If, as before, $A = [2(1-\nu)]/(1-2\nu)$ and $B = 1/(1-2\nu)$, the relevant derivatives comparable to (14)–(20) are in this case

$$\frac{\partial U}{\partial \eta_{11}} = \mu[1 - \lambda_1^{-B-1}\lambda_2^{-B+1}], \tag{56}$$

$$\frac{\partial U}{\partial \eta_{22}} = \mu[1 - \lambda_1^{-B+1}\lambda_2^{-B-1}], \tag{57}$$

$$\frac{\partial U}{\partial \eta_{12}} = 0, \tag{58}$$

$$\frac{\partial^2 U}{\partial \eta_{11}^2} = \mu A \lambda_2^{2-A}\lambda_1^{-2-A}, \tag{59}$$

$$\frac{\partial^2 U}{\partial \eta_{22}^2} = \mu A \lambda_2^{-2-A}\lambda_1^{2-A}, \tag{60}$$

$$\frac{\partial^2 U}{\partial \eta_{11}\,\partial \eta_{22}} = \mu(B-1)(\lambda_1\lambda_2)^{-A}, \tag{61}$$

$$\frac{\partial^2 U}{\partial \eta_{12}^2} = 4\mu(\lambda_1\lambda_2)^{-A}. \tag{62}$$

Hence the components γ_{ijkl} may be written as

$$\gamma_{1111} = \gamma_{2222} = \mu A(\lambda_1\lambda_2)^{-B} \tag{63}$$

$$\gamma_{1122} = \gamma_{2211} = \mu(B-1)(\lambda_1\lambda_2)^{-B}, \tag{64}$$

and

$$\gamma_{1212} = \gamma_{1221} = \gamma_{2112} = \gamma_{2121} = \mu(\lambda_1\lambda_2)^{-B}. \tag{65}$$

The constitutive equations are in this case

$$-p = \mu\left[\frac{\lambda_1}{\lambda_2} - (\lambda_1\lambda_2)^{-B}\right], \tag{66}$$

$$-q = \mu\left[\frac{\lambda_2}{\lambda_1} - (\lambda_1\lambda_2)^{-B}\right], \tag{67}$$

and the d.e.'s of neutral equilibrium are [cf. equations (30) and (31)]

$$u_{1,11}\left(\frac{\lambda_1}{\lambda_2}+B(\lambda_1\lambda_2)^{-B}\right)+u_{1,22}\left(\frac{\lambda_2}{\lambda_1}\right)+u_{2,12}B(\lambda_1\lambda_2)^{-B}=0, \tag{68}$$

$$u_{2,11}\left(\frac{\lambda_1}{\lambda_2}\right)+u_{2,22}\left(\frac{\lambda_2}{\lambda_1}+B(\lambda_1\lambda_2)^{-B}\right)+u_{1,12}B(\lambda_1\lambda_2)^{-B}=0. \tag{69}$$

The b.c.'s are in this case:

(i) *On edges* $x_2 = \pm\lambda_2 h/2$ [*cf. equations* (34) *and* (35)]

$$u_{1,2}\left(\frac{\lambda_2}{\lambda_1}\right)+u_{2,1}\left(\frac{\lambda_2}{\lambda_1}+q(1-\alpha)\right)=0, \tag{70}$$

$$u_{1,1}\left(B(\lambda_1\lambda_2)^{-B}-\frac{\lambda_2}{\lambda_1}\right)+u_{2,2}\left(B(\lambda_1\lambda_2)^{-B}+\frac{\lambda_2}{\lambda_1}\right)=0. \tag{71}$$

(ii) *On ends* $x_1 = 0, \lambda_1 l$ [*cf. equation* (38)]

As before the b.c.'s reduce to

$$u_{2,1}=u_1=0. \tag{72}$$

The d.e.'s (68), (69) can be solved using the same general trigonometric functions previously used for u_1, u_2, as defined in (39), (40). The solutions are identical to (43), (44) with the exception that K is now replaced by

$$\tilde{K}=[(1+\lambda_2^2\tilde{B}^2)/(1+\lambda_1^2\tilde{B}^2)]^{\frac{1}{2}}$$

where

$$\tilde{B}=B(\lambda_1\lambda_2)^{-A}.$$

The second b.c. (71) can be rewritten as,

$$u_{1,1}(\tilde{B}\lambda_1^2-1)+u_{2,2}(\tilde{B}\lambda_1^2+1)=0, \tag{73}$$

which is similar to (35).

It is not surprising, therefore, that the stability bounds for the Blatz–Ko material are given by expressions analogous to (53) and (54);

$$\frac{1}{\mu}(p-q)=\frac{2\lambda_2}{\lambda_1}-\left[\frac{2\tilde{K}\lambda_1\lambda_2\left(2+(1-\alpha)\frac{q}{\mu}\frac{\lambda_1}{\lambda_2}\right)}{\lambda_1^2+\lambda_2^2+(1-\alpha)\frac{q}{\mu}\lambda_1\lambda_2}\right]\tanh\left(\frac{n\pi h}{2l}\right)\coth\left(\frac{\tilde{K}n\pi h}{2l}\right) \tag{74}$$

for flexural instability and,

$$\frac{1}{\mu}(p-q)=\frac{2\lambda_2}{\lambda_1}-\left[\frac{2\tilde{K}\lambda_1\lambda_2\left(2+(1-\alpha)\frac{q}{\mu}\frac{\lambda_1}{\lambda_2}\right)}{\lambda_1^2+\lambda_2^2+(1-\alpha)\frac{q}{\mu}\lambda_1\lambda_2}\right]\coth\left(\frac{n\pi h}{2l}\right)\tanh\left(\frac{\tilde{K}n\pi h}{2l}\right) \tag{75}$$

for bulging instability.

6. THE INCOMPRESSIBLE NEO-HOOKEAN MATERIAL

Both materials considered reduce to the incompressible neo-Hookean material for $\nu = 0{\cdot}5, J_3 = 1$. It is therefore interesting to compare the stability bound obtained by letting $\nu \to 0{\cdot}5$ in the analysis of compressible materials presented above with those obtained by Wu and Widera using an incompressible analysis. It is worth noting that Wu and Widera's displacement field can be obtained directly from the application of the incompressibility condition to the solution field u_i given in equations (39) and (40). As the compressible material becomes incompressible then $\lambda_1 \to 1/\lambda_2$ and hence K and $\tilde{K}$ both become $1/\lambda_1^2$. Therefore the incompressible cases of (53) and (54), and (74) and (75) become

$$\frac{1}{\mu}(p-q) = \frac{2}{\lambda_1^2} - \left[\frac{2(2+(1-\alpha)(q/\mu)\lambda_1^2)}{\lambda_1^4+1+(1-\alpha)(q/\mu)\lambda_1^2}\right]\tanh\left(\frac{n\pi h}{2l}\right)\coth\left(\frac{n\pi h}{2l\lambda_1^2}\right) \tag{76}$$

for flexural stability, and

$$\frac{1}{\mu}(p-q) = \frac{2}{\lambda_1^2} - \left[\frac{2(2+(1-\alpha)(q/\mu)\lambda_1^2)}{1+\lambda_1^4+(1-\alpha)(q/\mu)\lambda_1^2}\right]\coth\left(\frac{n\pi h}{2l}\right)\tanh\left(\frac{n\pi h}{2l\lambda_1^2}\right) \tag{77}$$

for bulging instability. These are the correct solutions obtained by an "incompressible" analysis (Wu and Widera's solutions contain an erroneous numerical factor, which is pointed out in [8]).

7. NUMERICAL RESULTS

The best form of presentation for the stability bounds is not immediately obvious. The variables involved in specifying these bounds are p/μ, q/μ, h/l, ν and n, one of which will be dependent. In the previous work, [1–3, 5], families of characteristics of p/p_0 (where $p_0 = p|_{q=0}$) against q/p are shown for $n = 1$ at different aspect ratios h/l and different ν values (in [5] of course, $\nu = 0{\cdot}5$). The variables n and h/l are always multiplied together in the

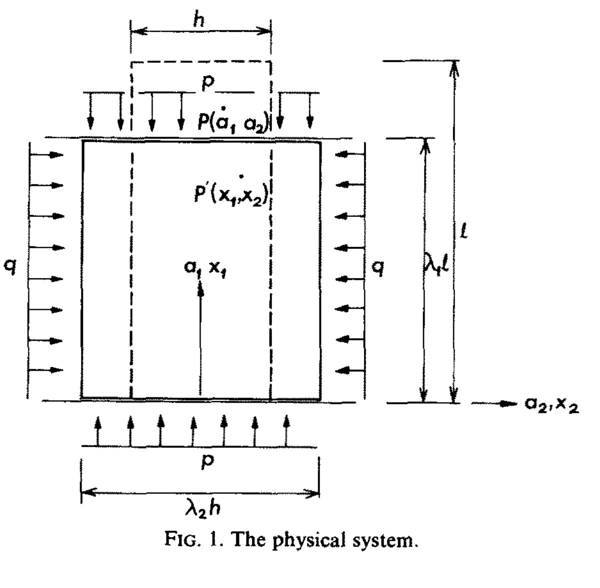

FIG. 1. The physical system.

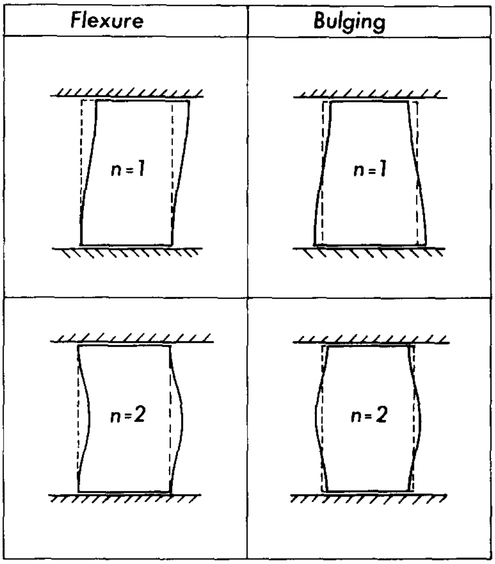

FIG. 2. Instability modes.

equations of stability bounds so that a change of n is equivalent simply to changing the product nh/l. It is therefore sufficient to consider $n = 1$, while varying h/l. Whether the case $n = 1$ is achievable in practice is dependent on the exact conditions at the end platens as pointed out by Levinson [14]. If the ends are pinned to the platens at their mid-points then the flexural mode of instability will only be achievable for even n, whereas if the platens are simply lubricated then $n = 1$ can be achieved for this mode (Fig. 2). Since [5] considers only the incompressible case, and [1–3] give p/p_0 against q/p characteristics for constant directional pressure which are independent of ν, it is desirable to present p/p_0 against q/p characteristics for $n = 1, \nu = 0{\cdot}5$ at various aspect ratios, for comparison with these previous results. It is also desirable to show examples of how these characteristics change with slight compressibility (i.e. as ν is gradually decreased from 0·5 by small amounts).

It is unfortunate that it seems impossible to show analytically (as Levinson did for a simpler problem in [14]) that the stability bounds for flexural instability are always "lower" than for bulging instability (i.e. for a fixed q, $p_{\text{flexural}} < p_{\text{bulging}}$), but it is observed from all present numerical results that this is the case. Hence, while bulging instability may have more than academic significance for certain cases, the flexural mode is practically the more important of the two. Figure 3 shows how the "Euler" end load p_0 (with $q = 0$) varies with aspect ratio h/l for both bulging and flexural instability. For the incompressible case it can be seen that the flexural instability is always associated with a lower p_0 than the bulging instability, and that the two critical pressures tend to merge as h/l increases. They are in fact almost indistinguishable from one another at $h/l = 2$. It is seen that a lowering of the ν value merely decreases the p_0 values, and that in this respect the Blatz–Ko material is less affected than the "polynomial" material.

Figure 4 corresponds to Fig. 2 of [5] and shows, for the incompressible case, a plot of p/p_0 against q/p for flexural instability at various aspect ratios. It is seen that hydrostatic q gives a single characteristic while constant-directional q gives a family of characteristics. Once again $h/l = 2$ gives a characteristic almost indistinguishable from that given by the

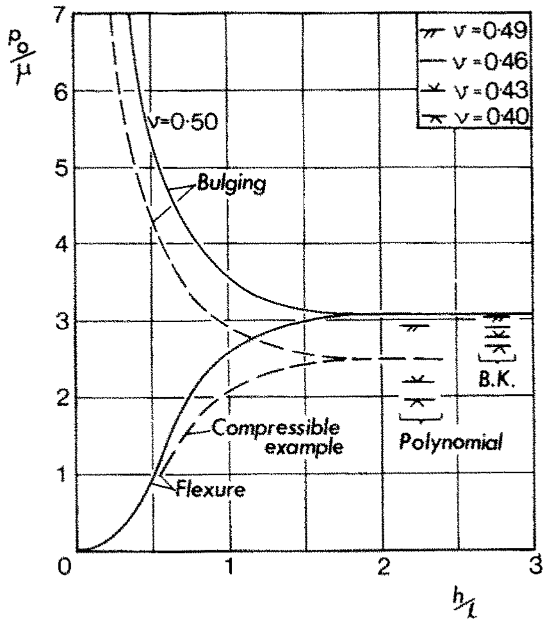

FIG. 3. Plot of "Euler" load–aspect ratio.

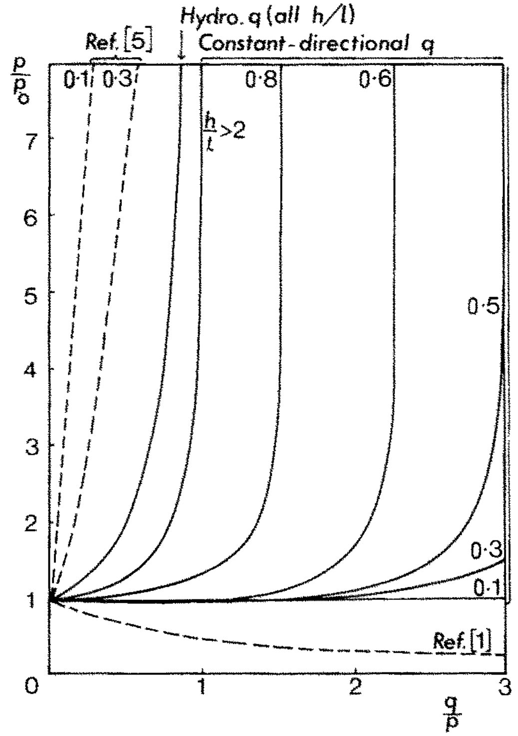

FIG. 4. Flexural instability in the incompressible case. Numbers on characteristics are values of h/l.

infinite pad, $h/l = \infty$. Successively lower aspect ratios show less stabilizing effect from q, continuing in the region $p/p_0 \simeq 1$ to high values of q/p. There is some very slight destabilizing effect on some of the characteristics, and the curve for $h/l = 0{\cdot}5$ does in fact reach a low of $p/p_0 = 0{\cdot}971$ at $q/p = 0{\cdot}865$. However, there appears to be no further destabilizing with increase of q/p, and on none of the other characteristics is there a drop of p/p_0 of the same order. It may be observed that correction of the small error in [5] does in fact cause a fairly considerable change in the qualitative aspects of the results in that:

(a) Constant-directional side-pressure has less stabilizing effect than hydrostatic side-pressure.

(b) Higher aspect ratios experience greater stabilization than lower aspect ratios.

Figure 5 is an example of the effect of slight compressibility on p/p_0 against q/p characteristics. The aspect ratio chosen is for the infinite pad (or $h/l > 2$). The changes in ν may be seen to cause a slight reduction in p/p_0 for the polynomial material and a very small increase for the Blatz–Ko material, but broadly the same qualitative behaviour is observed in each case as in the incompressible case. The slight destabilizing mentioned above is not enhanced by compressibility; in fact for $h/l = 0{\cdot}5$ the compressible characteristics lie successively slightly *above* the incompressible in the range of the destabilization, thus tending to reduce the effect. The effect is, in any case, not discernible on a graphical plot of the type of Fig. 5.

No plots are shown corresponding to 4 and 5 for the case of bulging instability. Since this mode of instability occurs, as far as one can tell, always after the flexural mode, it has

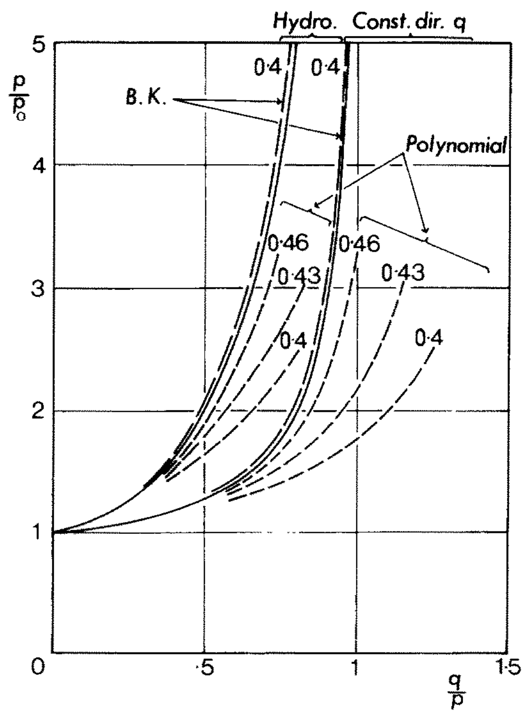

FIG. 5. Effect of reduction of ν on flexural instability. Numbers on characteristics are values of ν.

little physical relevance. There can also be little meaning to the term "stabilization" when applied to buckling from an unstable basic equilibrium state. It is worth emphasizing however that the transition from incompressible to compressible properties is once again smooth and gradual.

8. CONCLUSIONS

The "standard" material used in [1–3] has been noted in [9] to be inappropriate to deal with problems of large strains and to be meaningless for the incompressible case. It is not therefore surprising that the results given by it for constant-directional side pressure are not verified in any sense by the present analysis. On the other hand the analysis presented by Wu and Widera [5] for the incompressible case, apart from a slight numerical error which has affected their graphical results, is verified. The general conclusions on the problem are as follows.

1. That in general both hydrostatic and constant-directional side pressures stabilize the solid.

2. That the stabilizing effect is in all cases less marked for constant-directional than for hydrostatic side pressure.

3. That the *initial* stabilizing due to constant-directional pressure becomes less perceptible with decrease of the aspect ratio h/l and for a range of h/l a very slight destabilization occurs initially as q is increased from zero. However, as q increases, the ultimate effect is one of stabilization in all cases.

4. That the system's characteristics vary smoothly towards the incompressible characteristics as ν tends towards 0·5.

The ultimate conclusion, therefore, must be that the "standard" material has shown itself to be totally unsuitable for finite strain problems of this type. The results also indicate that an "incompressible" analysis using a realistic material model gives a good approximation to the behaviour of slightly compressible materials in such problems.

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Абстракт—Используя вариационный метод, исследуется неустойчивость прямоугольного тела, нагруженного в пиоском деформированном состоянии постоянными осевыми и поперечными давлениями. Осевые давления приложены посредством жестких пластин, боковые же давления могут быть гидростатические, либо постоянно направлённые. Исследуются два тмпа легко сжимаемого материала, похожего на резину. Оба сводятся к неогуковому материалу Ривлина, для несжимаемого случая. Презыдущие исследования, использующие "стандартный" материал и неогуковый материал, давали качественно разные результаты.

Оказывается, что стандартный материал непригоден для исследований таких же задач конечной деформации и что поведение исследоваемых, сжимающих материалов ведёт ровно к таким же из несжимаемого материала, когда коэффицинм Пуассона приближается к половине.